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WEAK SHOCKS AND STEADY WAVES IN A NONLINEAR ELASTIC ROD OR GRANULAR MATERIAL

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Abstract—Theories for a nonlinear elastic rod and a granular material are shown to be identical. Both cases admit shocks of two different types. Both also support steady waves, which may be either periodic or solitary waves.

INTRODUCTION AND EQUATIONS OF MOTION

A one dimensional continuum with one strain variable and one internal variable may be used to model either a long slender rod or longitudinal deformation in a granular material[1, 2]. In a rod the strain is the usual axial strain, and in a granular material it is the overall longitudinal strain. The internal variable measures radial strain in the rod and introduces radial forces and shears into the equations of motion. In the granular material it measures the volume fraction of solid material and introduces forces associated with the symmetrical expansion or collapse of pores. The internal variable also gives rise to an extra inertial effect, which in the rod is due to radial motion and in the granular material is due to pore dilation.

The equations that govern these two cases may be set down side by side for comparison. Only a purely mechanical version of either theory will be considered.

	Rod	*Granular material*	
Linear momentum	$S' = \rho\ddot{w}$	$\sigma' = \rho\dot{v}$	(1)
Micromomentum	$Q' - P = \frac{1}{2}\rho a^2\ddot{u}$	$h' + g = \rho k\ddot{\nu}$	(2)
Kinematics	$z = Z + w$	$v =$ velocity	(3)
	$\epsilon = w',\ v = \dot{w}$	$\epsilon =$ strain	
	$r = R(1+u)$ $q = u'$	$\nu =$ volume fraction of granules	
Stored energy	$W(u, q, \epsilon)$	$e(\nu, \nu', \epsilon)$	(4)
Stress potentials	$S = W_\epsilon$	$\sigma = e_\epsilon$	(5)
	$Q = W_q$	$h = e_{\nu'}$	
	$P = W_u$	$g = -e_\nu.$	

Here Z and t are independent variables for space and time, the prime and dot denote partial differentiation in the usual way, and subscripts denote differentiation with respect to the arguments of the function. In both cases the stored energy is an even function of its second argument. This latter fact has a profound effect on the types of waves that can occur.

Clearly the two continuum models are mathematically identical even though the physical interpretation of the various terms is dissimilar. In the rod the material point with cylindrical coordinates R, Z is located instantaneously in space at r, z so the independent kinematic variables are the axial displacement w and the radial strain u, which are assumed to depend only on Z, t. In the granular material v and ϵ correspond exactly to $\dot{w}$ and w', and the volume fraction ν plays the same role that u does for the rod. In the rod ρ is the reference density, and a is the initial radius. In the granular material ρ is also the reference density, but of the bulk material, not just the granules, and k is called the equilibrated inertia. In the rod S is the axial engineering stress, P is the sum of the average radial and circumferential stresses, and Q is the average radial moment of radial shear stress. In the granular material σ is the longitudinal stress, g is the intrinsic equilibrated body force, and h is the equilibrated stress. For further details the reader is referred to the original papers[1, 2], but since all stresses are derived from similar potentials, it is evident that, except for the signs of P and g, the analogy between the elastic rod and the granular material is complete.

On the basis of this analogy alone an important fact emerges. Since it is clear on physical grounds that the radius a is the natural length scale for all motions of the rod, *the quantity $\sqrt{2k}$ is the natural length scale for all motions in the granular material.* The remainder of this paper will use terminology appropriate for the rod model, but naturally *all results will apply equally well to the granular material.*

ONE DIMENSIONAL SHOCKS

The integral form of a general one dimensional conservation law may be written as follows.

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(Z, t)\,dZ = [fG]_{a(t)}^{b(t)} + [m]_{a(t)}^{b(t)} + \int_{a(t)}^{b(t)} s\,dZ. \tag{6}$$

The end points $a(t)$ and $b(t)$ are allowed to move arbitrarily with respect to the material, $f(Z, t)$ is the conserved quantity, $[\cdot]$ signifies the difference of the bracketed quantity between end points, G is the speed of the end point, m represents the flux of f added at the end points, and s is the interior supply of f. In words the equation simply says that the rate of change of some field quantity within an arbitrary interval is equal to the sum of convective transport through the ends of the interval, the flux of f at the ends, and the supply of f throughout the interior of the interval. In the usual way it is postulated that this equation holds even if there is a discontinuity in the field variables, say at a moving point with position $c(t)$ in the interior of the interval. Consequently, as is well known, the integral form of the balance law yields both the differential form and the jump conditions for shock waves

$$\dot{f} = m' + s \tag{7}$$

$$V[f] + [m] = 0 \tag{8}$$

where $V = \dot{c}(t)$ is the shock speed and $[\cdot] = (\cdot)^- - (\cdot)^+$ is the jump across $c(t)$.

Since it is reasonable to assume that the rod diameter cannot change abruptly, u must be continuous. Conservation of mass yields nothing new, and conservation of linear momentum with $f = \rho v$ and $m = S$ gives the familiar jump condition

$$\rho V[v] + [S] = 0, \tag{9}$$

but conservation of micromomentum with $f = \frac{1}{2}\rho a^2 \dot{u}$ and $m = Q$ yields an unfamiliar jump condition

$$\frac{1}{2}\rho a^2 V[\dot{u}] + [Q] = 0. \tag{10}$$

With the required compatibility conditions added

$$[v] + V[\epsilon] = 0, \qquad [\dot{u}] + V[q] = 0 \tag{11}$$

the jump conditions become

$$\rho V^2 = \frac{[S]}{[\epsilon]} \text{ and } \frac{1}{2}\rho a^2 V^2 = \frac{[Q]}{[q]}. \tag{12}$$

These conditions were given previously by Nunziato and Walsh[3]. In that same paper they also asserted that q (v_x in their terms) must be continuous in a strong shock wave. In the next section it will be shown that, in fact:

(i) There are two types of shock wave possible;

(ii) Although q may be continuous across one type, in general it is not; and

(iii) In the second type of shock wave q is always discontinuous, and although ϵ is usually discontinuous as well, $[\epsilon]$ will generally be weaker than $[q]$.

WEAK SHOCK WAVES

The jumps in the gradients of w and u may be expressed as follows:

$$\epsilon^- = \epsilon^+ + \alpha, \qquad q^- = q^+ + \beta. \tag{13}$$

With these expressions substituted in (12) the jump conditions become

$$\rho V^2\alpha = W_\epsilon(u^+, q^+ + \beta, \epsilon^+ + \alpha) - W_\epsilon(u^+, q^+, \epsilon^+) \tag{14}$$

$$\tfrac{1}{2}\rho a^2 V^2\beta = W_q(u^+, q^+ + \beta, \epsilon^+ + \alpha) - W_q(u^+, q^+, \epsilon^+). \tag{15}$$

Since these two equations have the three unknowns (α, β, V), it is to be expected that solutions will be one dimensional curves in a three dimensional space. The exact shape and location of these curves will depend parametrically on (u^+, q^+, ϵ^+). Clearly one branch is always given by the trivial case $\alpha = \beta = 0$ with V arbitrary, but bifurcations may occur where the Jacobian of eqns (14) and (15) taken with respect to α and β vanishes at $\alpha = \beta = 0$.

$$J(\alpha, \beta) = \begin{vmatrix} W_{\epsilon\epsilon} - \rho V^2 & W_{\epsilon q} \\ W_{\epsilon q} & W_{qq} - \frac{1}{2}\rho a^2 V^2 \end{vmatrix}. \tag{16}$$

$$J(0, 0) = 0. \tag{17}$$

Equation (17) determines the values of V at which bifurcations occur. In fact, it is just the characteristic condition for acceleration waves (see eqns (3.22)–(3.24) in [4]). To look for other branches near the bifurcation points, other than the trivial branch, first expand the energy W in powers of α and β in eqns (14) and (15). In the simplest case $u^+ = q^+ = \epsilon^+ = 0$. Bifurcations in (16) will appear at

$$\rho V^2 = W_{\epsilon\epsilon}(0, 0, 0) \text{ and } \tfrac{1}{2}\rho a^2 V^2 = W_{qq}(0, 0, 0). \tag{18}$$

Recall that W is an even function in q so that odd derivatives in q will not appear. The expansion now appears as follows.

$$\rho V^2\alpha = W^0_{\epsilon\epsilon}\alpha + \tfrac{1}{2}W^0_{\epsilon qq}\beta^2 + \tfrac{1}{2}W^0_{\epsilon\epsilon\epsilon}\alpha^2 + \tfrac{1}{2}W^0_{\epsilon\epsilon qq}\alpha\beta^2 + \tfrac{1}{6}W^0_{\epsilon\epsilon\epsilon\epsilon}\alpha^3 + \cdots \tag{19}$$

$$\tfrac{1}{2}\rho a^2 V^2\beta = W^0_{qq}\beta + W^0_{\epsilon qq}\alpha\beta + \tfrac{1}{6}W^0_{qqqq}\beta^3 + \tfrac{1}{2}W^0_{\epsilon\epsilon qq}\alpha^2\beta + \cdots \tag{20}$$

The superscript 0 indicates that the derivatives of W in (19) and (20) are to be evaluated at $u^+ = q^+ = \epsilon^+ = 0$. It is possible to regard either α or β as the primary variable.

Case (i). To examine the branch near the first of the bifurcations in (18) let β and V^2 be given in terms of α as a power series.

$$V^2 = V_0^2 + V_1^2\alpha + V_2^2\alpha^2 + \cdots \tag{21}$$

$$\beta = \beta_1\alpha + \beta_2\alpha^2 + \beta_3\alpha^3 + \cdots \tag{22}$$

Substitution of (21) and (22) into (19) and (20) and comparison of terms in lowest powers of α shows that

$$\beta_i = 0, \qquad i = 1, 2, 3, \ldots \tag{23}$$

$$\rho V_0^2 = W^0_{\epsilon\epsilon} \tag{24}$$

$$\rho V_1^2 = \tfrac{1}{2} W^0_{\epsilon\epsilon\epsilon} \tag{25}$$

$$\rho V_2^2 = \tfrac{1}{6} W^0_{\epsilon\epsilon\epsilon\epsilon}, \quad \text{etc.} \tag{26}$$

This is the case discussed by Nunziato and Walsh [3]. More generally, if $|aq^+| \ll 1$, but $\neq 0$, and $u^+ \neq 0$, $\epsilon^+ \neq 0$ as well, terms in all powers of α and β will appear throughout the expansions (19) and (20). If account is taken of the fact that odd derivatives of W with respect to q are odd in q, the net result is that (23)–(26) will all be modified by additional terms of order $O(aq^+)$ or higher. It is easily worked out that the lowest order corrections are

$$\rho V_0^2 = W^+_{\epsilon\epsilon} + O(a^2 q^{+2}) \tag{27}$$

$$\beta_1 = \frac{W^+_{qq\epsilon} q^+}{\frac{1}{2}a^2 W^+_{\epsilon\epsilon} - W^+_{qq}} + O(a^3 q^{+3}). \tag{28}$$

In (27) and (28) the superscript + indicates that the arguments of the derivatives of W are $(u^+, 0, \epsilon^+)$.

Case (ii). To examine the branch near the second of the bifurcations in (18) let α and V^2 be given in terms of β as a power series. Consider the simplest case first with $q^+ = 0$. It turns out that the series for α and V^2 only have even powers of β in this case.

$$\alpha = \alpha_2\beta^2 + \alpha_4\beta^4 + \cdots \tag{29}$$

$$V^2 = V_0^2 + V_2^2\beta^2 + V_4^2\beta^4 + \cdots \tag{30}$$

Substitution of (29) and (30) into (19) and (20) and comparison of equal powers of β gives the following.

$$\tfrac{1}{2}\rho a^2 V_0^2 = W^+_{qq} \tag{31}$$

$$\rho V_0^2 \alpha_2 = W^+_{\epsilon\epsilon}\alpha_2 + \tfrac{1}{2} W^+_{qq\epsilon} \tag{32}$$

$$\tfrac{1}{2}\rho a^2 V_2^2 = W^+_{qq\epsilon}\alpha_2 + \tfrac{1}{6} W^+_{qqqq} \tag{33}$$

$$\rho(\alpha_2 V_2^2 + \alpha_4 V_0^2) = W^+_{\epsilon\epsilon}\alpha_4 + \tfrac{1}{2} W^+_{\epsilon\epsilon\epsilon}\alpha_2^2 + \tfrac{1}{2} W^+_{qq\epsilon\epsilon}\alpha_2 + \tfrac{1}{24} W^+_{qqqq\epsilon} \tag{34}$$

$$\tfrac{1}{2}\rho a^2 V_4^2 = W^+_{qq\epsilon}\alpha_4 + \tfrac{1}{2} W^+_{qq\epsilon\epsilon}\alpha_2^2 + \tfrac{1}{120} W^+_{qqqqqq}. \tag{35}$$

As before the superscript + indicates that the arguments are $(u^+, 0, \epsilon^+)$. These may be solved sequentially for the α_i and the V_i^2. The first few terms are

$$V_0^2 = \frac{W^+_{qq}}{\frac{1}{2}a^2\rho} \tag{36}$$

$$\alpha_2 = \frac{\frac{1}{2}W^+_{\epsilon qq}}{\rho(c_2^2 - c_1^2)} \tag{37}$$

$$\tfrac{1}{2}\rho a^2 V_2^2 = \frac{\frac{1}{2}W^{+2}_{\epsilon qq}}{\rho(c_2^2 - c_1^2)} + \frac{1}{6}W^+_{qqqq} \tag{38}$$

where $c_1^2 = W^+_{\epsilon\epsilon}/\rho$, $c_2 = V_0$, as in (36) above. In the rod the speeds c_1 and c_2 are associated with bulk longitudinal and shear speeds[1]. In the granular material c_1 is associated with the compressibility of the granules without voids, and c_2 with changes in the volume fraction without compression of the granules[4]. Ordinarily $c_1 > c_2$, but that is not necessarily true for all values of u^+ and ϵ^+. The point is that both V_2^2 and V_4^2 may be either positive or negative depending on material properties and the strain state ahead of the wave.

As before a more general case occurs if $0 < |aq^+| \ll 1$. Then the expansions in (29) and (30) will contain all powers of β. The new coefficients, that is, the coefficients of the odd powers of β in the equations that replace (29) and (30), will all be of order $O(aq^+)$ and the ones already calculated from (31) to (35), that is, the coefficients of the even powers of β, will change only by order $O(a^2q^{+2})$. This result again follows from the fact that W is even in q.

SHOCK STABILITY

Shock speeds have been calculated above simply as possible solutions of (14) and (15). It has been convenient to give the speeds parametrically in powers of the amplitude, using either α or β as appropriate for the type of wave. However, it is not reasonable to accept all values of shock speed computed in this manner without some consideration of shock stability. Since it is not possible to appeal to a thermodynamic argument in a purely mechanical theory such as this, it is necessary to adopt some other criterion such as the Lax stability criterion[5] whereby it is required that the characteristic speed behind (ahead of) the shock of the same family is greater than (less than) the shock speed. Clearly a necessary condition is that the shock speed for nonzero amplitude must be greater than the bifurcation speed. This condition is met if the shock speed is a monotonically increasing function of the amplitude, but without detailed calculation, it does not seem possible to say whether the characteristic speed behind the shock will be greater than the shock speed. Some of the possibilities that may occur are shown in Figs. 1–5.

Figure 1 shows the case discussed in [3]. With $q^+ = 0$, the solid line shows the common example of compressive shocks. The dashed line is the continuation of the solution to (14) and (15), but now this branch is not allowed on stability grounds as discussed above. If $q^+ \neq 0$, the picture changes only slightly with the interrupted line showing allowed shocks and the dotted line showing forbidden shocks. Figure 1(b) shows that $\beta \equiv 0$ in the simplest case, but that in general $\beta \neq 0$ when $q^+ \neq 0$.

The possibilities for the second type of shock are much richer. Some of these are shown in Figs. 2–5 where it has been assumed throughout that $c_1 \neq c_2$. Figure 2 shows the case

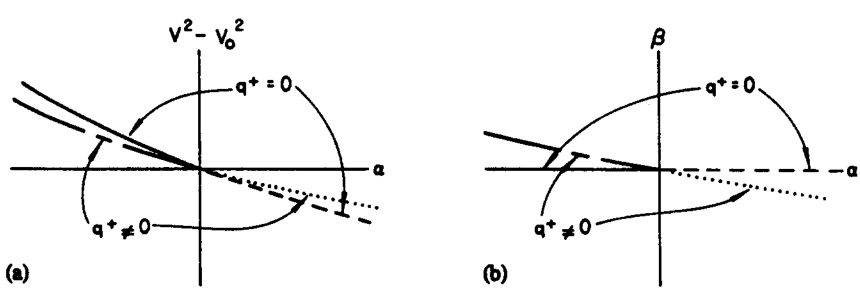

Fig. 1. (a) Sketch of shock speed vs amplitude and (b) jump in q vs amplitude for quasi-longitudinal shocks. The solid and interrupted lines show allowed shocks for $q^+ = 0$ and $q^+ \neq 0$ respectively. The dashed and dotted lines show forbidden shocks for the same cases.

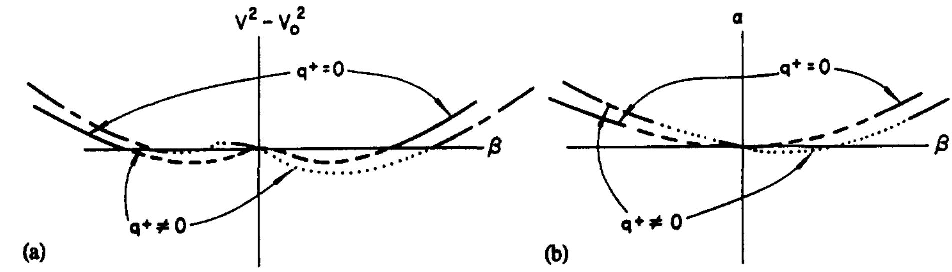

Fig. 2. (a) Sketch of shock speed vs amplitude and (b) jump in ϵ vs amplitude for radial (void volume) shocks. $V_2^2 < 0$, $V_4^2 > 0$.

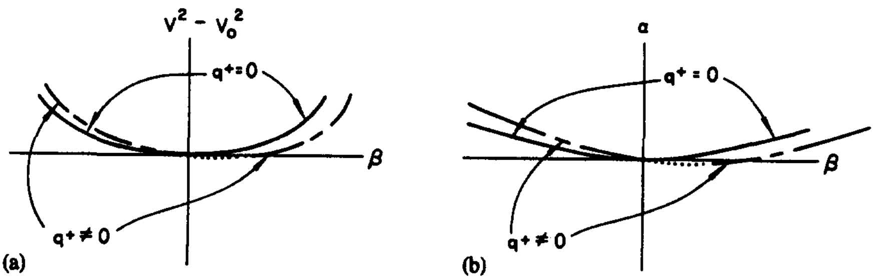

Fig. 3. (a) Sketch of shock speed vs amplitude and (b) jump in ϵ vs amplitude for radial (void volume) shocks. $V_2^2 \geqslant 0$, $V_4^2 > 0$.

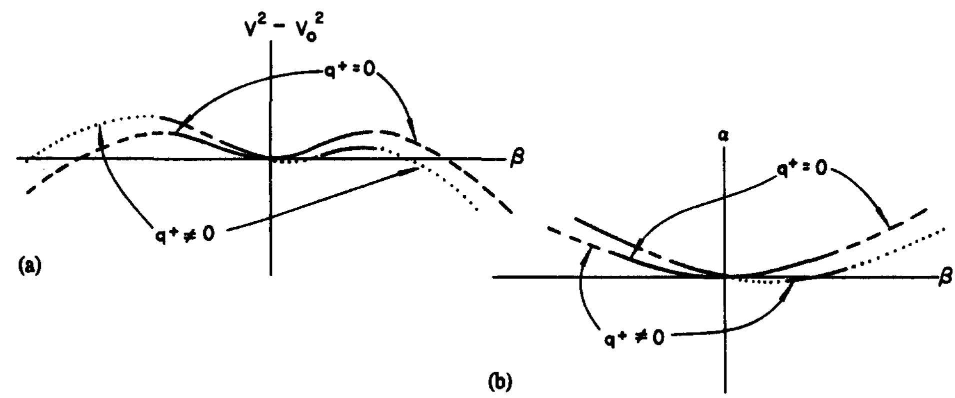

Fig. 4. (a) Sketch of shock speed vs amplitude and (b) jump in ϵ vs amplitude for radial (void volume) shocks. $V_2^2 > 0$, $V_4^2 < 0$.

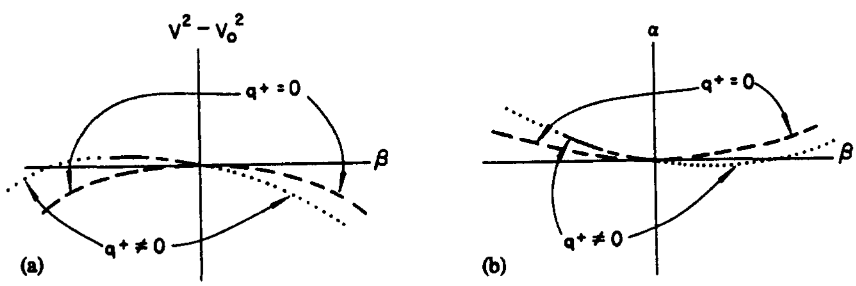

Fig. 5. (a) Sketch of shock speed vs amplitude and (b) jump in ϵ vs amplitude for radial (void volume) shocks. $V_2^2 \leqslant 0$, $V_4^2 < 0$.

when $V_2^2 < 0$ and $V_4^2 > 0$. If $q^+ = 0$, no shocks at all are possible unless the amplitude is greater than a critical value and then the amplitude may be of either sign. If $q^+ \neq 0$, then shocks of one sign only are possible for a limited range of amplitude, but for large enough amplitudes shocks of either sign are possible. The fact that $q^+ \neq 0$ has merely introduced some asymmetry into the curve. Likewise the curve for α as a function of β

is symmetric if $q^+ = 0$ and asymmetric if $q^+ \neq 0$. Figures 3–5 show the other possibilities. Note that in every case stable shocks may exist for some range of amplitude if $q^+ \neq 0$, and in nearly every case if $q^+ = 0$.

STEADY WAVES

Steady waves in a nonlinear elastic rod have been discussed previously[1] for the special case when the stored energy function can be decomposed into $W(u, q, \epsilon) = W_1(u, \epsilon) + W_2(q)$. Here the analysis for steady waves will be briefly reviewed, but for an arbitrary stored energy. In a steady wave all field variables depend only on a special combination of Z and t, namely $\xi = Z - ct$ where c is a constant speed of propagation. The partial differential equations (1) and (2) become ordinary differential equations, and it was shown in [1] that the solution can be reduced to quadratures of the following form,

$$\epsilon = f(u; A, B, c) \tag{39}$$

$$u'^2 = g(u; A, B, c) \tag{40}$$

where the functions f and g satisfy two integrals of the motion identically. These are

$$F(u, g, f) = W_f - \rho c^2 f - A = 0 \tag{41}$$

$$G(u, g, f) = 2W_g g + Af - W - \tfrac{1}{4}\rho c^2 a^2 g + \tfrac{1}{2}\rho c^2 f^2 - B = 0 \tag{42}$$

where A and B are constants of integration. A power series solution for f and g in terms of u may be found by differentiating (41) and (42) with respect to u and solving each successive pair of equations for the derivatives of f and g. The choice of initial values for f, g and u determines the constants A and B. For example, choosing $f(0) = g(0) = 0$ leads to values of A, B, and the derivatives at $u = 0$ as follows.

$$A = B = 0 \tag{43}$$

$$f_u = -\frac{W_{fu}}{W_{ff} - \rho c^2} \tag{44}$$

$$g_u = 0 \tag{45}$$

$$f_{uu} = -\frac{(W_g - \tfrac{1}{4}\rho c^2 a^2)(W_{uuf} + 2W_{uff} f_u + W_{fff} f_u^2) + W_{fg}(W_{uu} + W_{uf} f_u)}{(W_g - \tfrac{1}{4}\rho c^2 a^2)(W_{ff} - \rho c^2)} \tag{46}$$

$$g_{uu} = \frac{W_{uu}\left\{\dfrac{W_{uu}W_{ff} - W_{uf}^2}{W_{uu}} - \rho c^2\right\}}{(W_{ff} - \rho c^2)(W_g - \tfrac{1}{4}\rho c^2 a^2)} \tag{47}$$

$$g_{uuu} = \frac{-(4W_{fg} f_u + 3W_{ug}) g_{uu} + W_{uuu} + 2W_{uuf} f_u + W_{uff} f_u^2 + W_{uf} f_{uu}}{(W_g - \tfrac{1}{4}\rho c^2 a^2)}. \tag{48}$$

Truncated versions of (39) and (40) are now

$$\epsilon = f_u u + \tfrac{1}{2} f_{uu} u^2 \tag{49}$$

$$u'^2 = \tfrac{1}{2} g_{uu} u^2 + \tfrac{1}{6} g_{uuu} u^3. \tag{50}$$

The coefficients f_u and g_{uu} are identical to those in [1], and the other two coefficients are only slightly modified so the behavior of solutions is still the same as described in [1]. Thus if c is chosen such that $g_{uu} > 0$, solutions are solitary waves.

$$u = -\frac{3g_{uu}}{g_{uuu}}\operatorname{sech}^2\left(\frac{\sqrt{g_{uu}}\xi}{2^{3/2}}\right). \tag{51}$$

Clearly c must be chosen such that $|3g_{uu}| < |g_{uuu}|$. Propagating bulges (necks) occur if $g_{uuu} < 0$ ($g_{uuu} > 0$). The situation for a propagating bulge is shown in Fig. 6.

In terms of the elastic moduli of linear elasticity, the second derivatives of W used above may be written (see [1])

$$W_{ff} = \lambda + 2\mu, \quad W_{uu} = 4(\lambda + \mu), \quad W_{uf} = 2\lambda, \quad W_g = 1/4a^2\mu \tag{52}$$

where λ and μ are the Lamé elastic constants. Therefore

$$g_{uu} = \frac{16(\lambda + \mu)(E - \rho c^2)}{a^2(\lambda + 2\mu - \rho c^2)(\mu - \rho c^2)} \tag{53}$$

where E is Young's modulus. The conditions on c will be satisfied if $0 < (\rho c^2/E) - 1 \ll 1$.

If $g(0) = u_0'^2 \neq 0$, then (50) must be replaced by

$$u'^2 = u_0'^2 + g_u u + \tfrac{1}{2}g_{uu}u^2 + \tfrac{1}{6}g_{uuu}u^3. \tag{54}$$

In this equation it turns out that if $|au_0'| = \delta \ll 1$, then $g_u = O(\delta^2)$, and both g_{uu} and g_{uuu} differ from their previous values by terms that are $O(\delta^2)$. If c is chosen so that g_{uu} is negative, then solutions to (54) are periodic. The situation that is envisioned here is shown in Fig. 7. Solutions will oscillate between the points u_1 and u_2, and may be expressed in terms of elliptic integrals (see [6], Chap. 17, for the appropriate transformations). Because of the restriction that $g_{uu} < 0$, not all speeds are possible for periodic waves. In the linear limit this gives the familiar conditions $\mu < \rho c^2 < E$ or $\lambda + 2\mu < \rho c^2$.

SUMMARY AND DISCUSSION

It has been shown that there is a complete mathematical analogy between theories for a nonlinear rod and a granular material. Since rods are familiar objects to applied

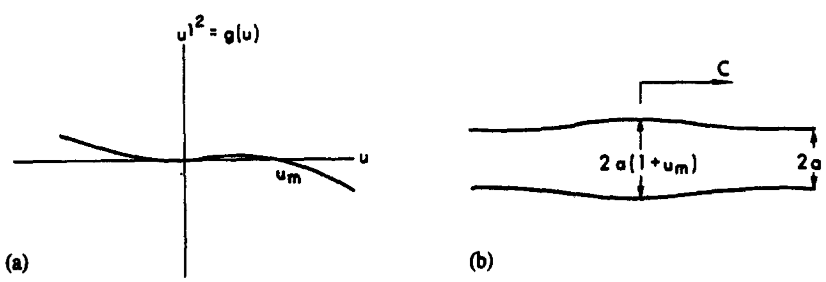

Fig. 6. (a) Sketch of $g(u)$ vs u for $g(0) = 0$, $g_{uu} > 0$ and $g_{uuu} < 0$, and (b) solitary wave corresponding to 6(a).

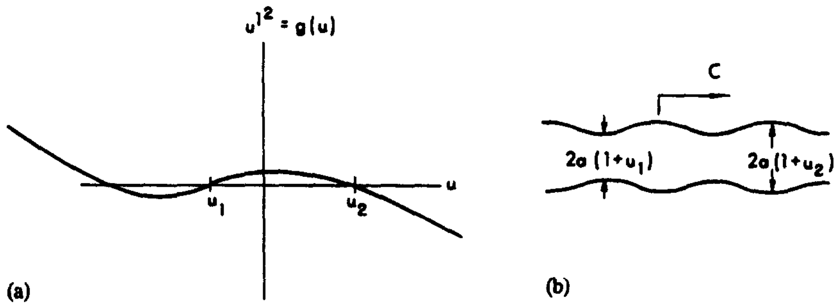

Fig. 7. (a) Sketch of $g(u)$ vs u for $g_u = 0$, $g_{uu} < 0$ and $g_{uuu} < 0$, and (b) periodic wave corresponding to 7(a).

mechanicians, but study of granular materials is of relatively recent origin, the analogy should be useful. For example, it was noted that a characteristic length of $\sqrt{2k}$ could be identified for a granular material, but even more important any statement about the dynamics or deformation of a rod has an exact counterpart for a granular material. Thus, for static homogeneous deformations of a granular material the equation of balance for micromomentum (2) reduces to $g(\nu, 0, \epsilon) = 0$, which defines an equilibrium relation $\nu = \hat{\nu}(\epsilon)$, and the negative of the slope of this relation defines the analog of Poisson's ratio. Furthermore, along this equilibrium line there will be a modified stress–strain relation $\sigma = \hat{\sigma}(\epsilon)$, the slope of which corresponds to Young's modulus E. In fact, it is Young's modulus that appears as the first term in the numerator of (47). For the granular material the analog modulus may be written

$$\frac{e_{\nu\nu}e_{\epsilon\epsilon} - e_{\nu\epsilon}^2}{e_{\nu\nu}}.$$

Since it is known that Young's modulus has fundamental importance for rod deformations, the analog modulus must have fundamental importance for deformations in granular materials. In particular, for low frequency disturbances with wavelengths substantially longer than the characteristic length, it is to be expected that the analog of Young's modulus will dominate.

It has been pointed out that both rods and granular materials support two kinds of shock wave. One is the familiar longitudinal shock for both cases, but the second is somewhat unfamiliar. The second type of shock is associated with the micromomentum and is accompanied by a discontinuity in the gradient of radial strain for the rod or by a discontinuity in the gradient of volume fraction for the granular material. In the rod such a discontinuity is associated with a radially symmetric shearing motion so it can be readily visualized, and it is not surprising that the speed is controlled by the shear modulus, $W_{qq}/\frac{1}{2}a^2$. The analog for the granular material, however, is totally unfamiliar, but may be of some importance. The fact that the stored energy is even in q (or ν') has also been pointed out. The symmetry of the amplitude curves for $q^+ = 0$ in Figs. 2–5 and the slight biasing to one side or the other for $q^+ \neq 0$ are direct consequences of that fact.

Finally it has been pointed out that finite amplitude periodic waves and solitary waves can propagate in either a rod or a granular material. There are two spectral branches for wave speeds, as was noted at the end of the last section. Detailed calculations with the elliptic integrals that satisfy (54) would show the effect of finite amplitude on wave speed for periodic wavetrains with a fixed wavelength.

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